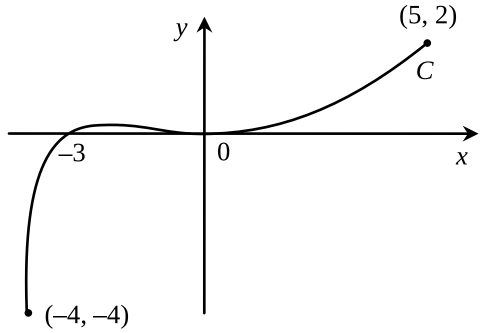
H2 MATHEMATICS 9740
PAPER 2 SOLUTIONS

1 **(i)**



(ii)

$$\frac{dx}{dt} = 2t + 4$$

$$\frac{dy}{dt} = 3t^2 + 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 2t}{2t + 4}$$

When $t = 2$, $x = 2^2 + 4(2) = 12$, $y = 2^3 + 2^2 = 12$

$$\frac{dy}{dx} = \frac{3(2)^2 + 2(2)}{2(2) + 4} = \frac{16}{8} = 2$$

$y = 2(x - 12) + 12$

$\therefore$ Cartesian equation of l is $y = 2x - 12$

(iii)

$y = 12x - 12$

$t^3 + t^2 = 2(t^2 + 4t) - 12$

$t^3 + t^2 = 2t^2 + 8t - 12$

$t^3 - t^2 - 8t + 12 = 0$

$(t - 2)(t^2 + t - 6) = 0$

$(t - 2)^2 (t + 3) = 0$

$t = 2$ (rej.) or $t = -3$

$x = (-3)^2 + 4(-3) = -3$

$y = (-3)^3 + (-3)^2 = -18$

Coordinates of Q is $(-3, -18)$

2 **(i)**

$$\overrightarrow{OP} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB}$$

$$= \frac{1}{3}\begin{pmatrix}14\\14\\14\end{pmatrix} + \frac{2}{3}\begin{pmatrix}11\\-13\\2\end{pmatrix}$$

$$= \begin{pmatrix}12\\-4\\6\end{pmatrix}$$

Coordinates of P is $(12, -4, 6)$

(ii)

$$\overrightarrow{OP} \cdot \overrightarrow{AB} = \overrightarrow{OP} \cdot (\overrightarrow{OB} - \overrightarrow{OA})$$

$$= \begin{pmatrix}12\\-4\\6\end{pmatrix} \cdot \left[\begin{pmatrix}11\\-13\\2\end{pmatrix} - \begin{pmatrix}14\\14\\14\end{pmatrix}\right] = \begin{pmatrix}12\\-4\\6\end{pmatrix} \cdot \begin{pmatrix}-3\\-27\\-12\end{pmatrix}$$

$$= (12)(-3) + (-4)(-27) + (6)(-12)$$

$$= 0$$

Since $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$, OP and AB are perpendicular.

(iii)

$$\mathbf{c} = \frac{1}{\sqrt{12^2 + (-4)^2 + 6^2}}\begin{pmatrix}12\\-4\\6\end{pmatrix} = \frac{1}{14}\begin{pmatrix}12\\-4\\6\end{pmatrix}$$

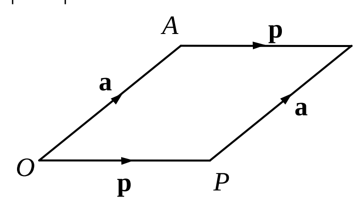
$|\mathbf{a} \cdot \mathbf{c}|$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$.

(iv)

$$\mathbf{a} \times \mathbf{p} = \begin{pmatrix}14\\14\\14\end{pmatrix} \times \begin{pmatrix}12\\-4\\6\end{pmatrix} = \begin{pmatrix}(14)(6) - (14)(-4)\\(14)(12) - (14)(6)\\(14)(-4) - (14)(12)\end{pmatrix}$$

$$= \begin{pmatrix}140\\84\\-224\end{pmatrix}$$

$|\mathbf{a} \times \mathbf{p}|$ is the area of the parallelogram:



Area of triangle $OAP = \frac{1}{2}|\mathbf{a} \times \mathbf{p}|$

$$= \frac{1}{2}\left|\begin{pmatrix}140\\84\\-224\end{pmatrix}\right| = \frac{1}{2}\sqrt{140^2 + 84^2 + (-224)^2}$$

$$= \frac{1}{2}\sqrt{76832}$$

$= 98\sqrt{2}$ square units

3 **(i)**

Let $f(x) = y$, then $f^{-1}(y) = x$.

$$y = \frac{ax}{bx - a}$$

$\Leftrightarrow bxy - ay = ax \Leftrightarrow bxy - ax = ay$

$$\Leftrightarrow x = \frac{ay}{by - a} = f^{-1}(y)$$

$$\therefore f^{-1}(x) = \frac{ax}{bx - a}, x \in \mathbb{R}, x \neq \frac{a}{b}$$

Since $f(x) = f^{-1}(x)$, $f^2(x) = ff(x) = f^{-1}f(x) = x$

Range of $f^2 = \mathbb{R} \setminus \{\frac{a}{b}\}$

(ii) $g(x)=\frac{1}{x}, x\in\mathbb{R}, x\neq 0$

Range of $g=\mathbb{R}\setminus\{0\}$

Domain of $f=\mathbb{R}\setminus\{\frac{a}{b}\}$

The domain of f does not exclude the set containing zero, since a, b are non-zero.

Since range of $g\not\subseteq$ domain of f, the composite function fg does not exist.

(iii) $f^{-1}(x)=x$

$\frac{ax}{bx-a}=x$

$ax=bx^2-ax$

$bx^2-2ax=0$

$x(bx-2a)=0$

$x=0$ or $\frac{2a}{b}$

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4 **(i)** $\frac{d^2n}{dt^2}=10-6t$

$\frac{dn}{dt}=\int 10-6t\,dt=10t-3t^2+c$

$n=\int 10t-3t^2+c\,dt=5t^2-t^3+ct+b$

When $t=0, n=100\Rightarrow b=100$

$\therefore n=5t^2-t^3+ct+100$, where c is an arbitrary constant.

For the family of solution curves, choose a positive, zero and negative value for c.

(ii) $\frac{dn}{dt}=3-0.02n$

$\int\frac{1}{3-0.02n}\,dn=\int dt$

$-\frac{1}{0.02}\ln|3-0.02n|=t+c$

$\ln|3-0.02n|=-0.02(t+c)$

$3-0.02n=e^{-0.02t-0.02c}$

$0.02n=3-e^{-0.02t-0.02c}$

$n=50(3-Ae^{-0.02t})$, where $A=e^{-0.02c}$

When $t=0, 100=50(3-e^{-0.02c})=50(3-A)$

$100=150-50A\Rightarrow A=1$

$\therefore n=50(3-e^{-0.02t})$

The population size will eventually remain constant at 150 000.

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5 To obtain a quota sample of size 100, divide the population, i.e. the cinema-goers, into mutually exclusive sub-groups, i.e. male and females. Select the first 50 males and first 50 females which enter the cinema for the survey.

A disadvantage of quota sampling is that the data may be biased as the surveyor may approach people who appear to be more willing to be surveyed.

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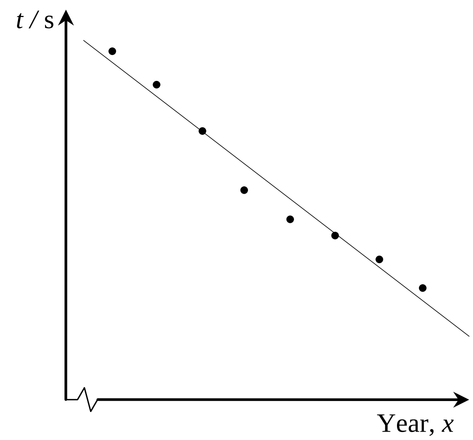
6 **(i)**



(ii) The linear model is appropriate since most of the points of the bivariate lie close to the least squares regression line, where r = –0.986 is close to –1. Since the world record time is expected to decrease over time as technology is developed to help runners improve their timing, a linear model can be used.

(iii) A quadratic model is not suitable for long-term predictions as there will be a point in time where the value of t increases as x increases. This is not an appropriate model since the world record time cannot be higher than before.

(iv)

x	1930	1940	1950	1960	1970	1980	1990	2000
t	40.4	36.4	31.3	24.5	21.1	19	16.3	13.1
$\ln t$	3.7	3.59	3.44	3.2	3.05	2.94	2.79	2.57

From GC, $\ln t=34.853-0.016128x$

When $x=2010$,

$\ln t=34.853-0.016128(2010)=2.4357$

$t=e^{2.4357}=11.424\approx 11.4$

The world record time as at 1st January 2010 is 3 minutes 41.4 seconds.

The prediction is unreliable as we are extrapolating the data beyond the year 2000.

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7 **(i)** P(a randomly chosen component is faulty)

$=\frac{25}{100}\times 0.05+\frac{75}{100}\times 0.03=\frac{7}{200}$

(ii) f(p) = P(component supplied by A | it is faulty)

$$=\frac{\frac{p}{100}\times 0.05}{\frac{p}{100}\times 0.05+\frac{100-p}{100}\times 0.03}$$

$$=\frac{0.05p}{0.05p+3-0.03p}$$

$$=\frac{0.05p}{0.02p+3}$$

$$f'(p)=\frac{0.05(0.02p+3)-0.05p(0.02)}{(0.02p+3)^2}$$

$$=\frac{3}{20(0.02p+3)^2}>0$$

When $0\le p\le 100$, f is an increasing function since $f'(p)>0$.

When p increases, f(p) increases. When the company buys a larger percentage of its electronic components from supplier A, the probability that a randomly chosen faulty component is supplied by A increases.

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8 **(i)** $\text{Number of ways}=\frac{8!}{3!}=6720$

(ii) $\text{Number of ways}=\frac{6!}{3!}\times {}^7P_2=5040$

(iii) $\text{Number of ways}=2\times 4!\times\frac{4!}{3!}=192$

(iv)

									No. of ways
Case 1	E			E			E		5!
Case 2	E			E				E	5!
Case 3	E				E			E	5!
Case 4		E			E			E	5!

$\text{Number of ways}=4\times 5!=480$

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9 **(i)** $M\sim N(2.5, 0.1^2)$

$\overline{M}\sim N\left(2.5,\frac{0.1^2}{n}\right)$

Let $Z=\frac{\overline{M}-2.5}{0.1/\sqrt{n}}$, where $Z\sim N(0,1)$.

$P(\overline{M}>2.53)=0.0668$

$P\left(Z>\frac{2.53-2.5}{0.1/\sqrt{n}}\right)=1-0.0668=0.9332$

$Z=1.5001=\frac{2.53-2.5}{0.1/\sqrt{n}}$

$n=\left(\frac{(0.1)(1.5001)}{2.53-2.5}\right)^2=25.002\approx 25$

(ii) $M_1+M_2+\ldots+M_{21}\sim N(21\times 2.5, 21\times 0.1^2)$

$S_1+S_2+\ldots+S_{24}\sim N(24\times 2.0, 24\times 0.08^2)$

$M_1+M_2+\ldots+M_{21}+S_1+S_2+\ldots+S_{24}$

$\sim N(21\times 2.5+24\times 2.0, 21\times 0.1^2+24\times 0.08^2)$

$P(M_1+M_2+\ldots+M_{21}+S_1+S_2+\ldots+S_{24}<100)$

$=0.203496\approx 0.203$

(iii) $3M\sim N(3\times 2.5, 3^2\times 0.1^2)$

$S_1+S_2+S_3+S_4\sim N(4\times 2.0, 4\times 0.08^2)$

$S_1+S_2+S_3+S_4-3M$

$\sim N(4\times 2.0-3\times 2.5, 4\times 0.08^2+3^2\times 0.1^2)$

$P(S_1+S_2+S_3+S_4<3M)$

$P(S_1+S_2+S_3+S_4-3M<0)$

$=0.070701\approx 0.0707$

(iv) Assume that the thickness of the books is independent of one another.

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10 **(i)** $\bar{x}=\frac{\Sigma x}{n}=\frac{86.4}{9}=9.6$

$s^2=\frac{1}{n-1}\left(\Sigma x^2-\frac{(\Sigma x)^2}{n}\right)=\frac{1}{8}\left(835.92-\frac{86.4^2}{9}\right)$

$=0.81$

The unbiased estimates of the mean and variance of X is 9.6 and 0.81 respectively.

(ii) Assume that the mean mass of sugar in a packet follows a normal distribution.

$H_0:\mu=10\quad\text{vs}\quad H_1:\mu\ne 10$

Perform a 2-tail test at the 5% level of significance.

Under H_0, $\overline{X}\sim N\left(\mu_0,\frac{\sigma^2}{n}\right)$, $T=\frac{\overline{X}-\mu_0}{s/\sqrt{n}}$,

where, $\mu_0=10$, $\bar{x}=9.6$, $s=\sqrt{0.81}$, $n=9$

Using a t-test, p-value $=0.219$

Since the p-value = 0.219 > 0.05, we do not reject H_0 and conclude that there is insufficient evidence at the 5% level of significance that the claim is incorrect.

The Central Limit Theorem does not apply in this context as n = 9 not sufficiently large. This Theorem only applies for large values of n, i.e. n > 50.

(iii) The z-test is used instead of the t-test, since the population variance of X is known.

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11 **(i)** Firstly, assume that there are two mutually exclusive outcomes, i.e. the car is either red or not red.

Secondly, assume that the probability remains the same for each trial, i.e. the probability of a red car observed is the same for each trial.

(ii) $R\sim B(20, 0.15)$

$P(4\le R<8)$

$=P(4\le R\le 7)$

$=P(R\le 7)-P(R\le 3)$

$=0.34635$

≈ 0.346

(iii) $np = 240 \times 0.3 = 72$

$n(1-p) = 240 \times (1-0.3) = 168$

Since $np > 5$ and $n(1-p) > 5$,

$R \sim \mathrm{N}(72, 50.4)$ approximately.

$\mathrm{P}(R < 60)$

$= \mathrm{P}(R < 59.5)$ (by continuity correction)

$= 0.039141$

≈ 0.0391

(iv) $np = 240 \times 0.02 = 4.8$

Since $np < 5$,

$R \sim \mathrm{Po}(4.8)$ approximately.

$\mathrm{P}(R = 3)$

$= 0.15169$

≈ 0.1517 (4 d.p.)

This approximation is appropriate in this case as $np = 4.8 < 5$ is sufficiently small and $n = 240 > 50$ is sufficiently large. Note that the actual value calculated when $R \sim \mathrm{B}(240, 0.02)$ is 0.1516, which is close to 0.1517.

(v) $\mathrm{P}(R = 0) + \mathrm{P}(R = 1)$

$= \binom{20}{0} p^0 (1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19}$

$= (1-p)^{20} + 20p(1-p)^{19}$

$= 0.2$

Using GC, $p = 0.142$
