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# H2 MATHEMATICS 9740
# PAPER 1 SOLUTIONS

**1** **(i)** Let $u_n = An^2 + Bn + C$, where A, B and C are constants.

$u_1 = A + B + C = 10$

$u_2 = 4A + 2B + C = 6$

$u_3 = 9A + 3B + C = 5$

Solving the simultaneous equations by GC,

$$a = \frac{3}{2}, b = -\frac{17}{2}, c = 17 \Rightarrow u_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17$$

**(ii)**

$$u_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17 > 100$$

$$\frac{3}{2}n^2 - \frac{17}{2}n + 17 - 100 > 0$$

By GC, $n = 10.793$ or $n = -5.13$

Since $n > 0$, the set of values of $n$ for which $u_n > 100$ is $\{n : n \geq 11, n \in \mathbb{Z}\}$

**2**

$$\int_0^1 \frac{1}{4 - x^2}\,dx = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1 - p^2x^2}}\,dx$$

$$\frac{1}{4}\left[\ln\left|\frac{2+x}{2-x}\right|\right]_0^1 = \frac{1}{p}\left[\sin^{-1}(px)\right]_0^{\frac{1}{2p}}$$

$$\frac{1}{4}\ln 3 = \frac{1}{p} \times \frac{\pi}{6}$$

$$p = \frac{2\pi}{3\ln 3}$$

**3** **(i)**

$$\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$

$$= \frac{(n)(n+1) - 2(n-1)(n+1) + (n-1)(n)}{(n-1)(n)(n+1)}$$

$$= \frac{n^2 + n - 2n^2 + 2 + n^2 - n}{n(n^2 - 1^2)} = \frac{2}{n^3 - n}, \text{ where } A = 2$$

**(ii)**

$$\sum_{r=2}^{n} \frac{1}{r^3 - r} = \frac{1}{2}\sum_{r=2}^{n} \frac{2}{r^3 - r} = \frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right)$$

$$= \frac{1}{2}\left(\begin{array}{cccccc} & \frac{1}{1} & - & \frac{2}{2} & + & \frac{1}{3} \\ + & \frac{1}{2} & - & \frac{2}{3} & + & \frac{1}{4} \\ + & \frac{1}{3} & - & \frac{2}{4} & + & \frac{1}{5} \\ & \vdots & & & & \\ + & \frac{1}{n-3} & - & \frac{2}{n-2} & + & \frac{1}{n-1} \\ + & \frac{1}{n-2} & - & \frac{2}{n-1} & + & \frac{1}{n} \\ + & \frac{1}{n-1} & - & \frac{2}{n} & + & \frac{1}{n+1} \end{array}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

**(iii)** The sequence converges as $n \to \infty$, $\frac{1}{n} \to 0$, $\frac{1}{n+1} \to 0$ and $\frac{1}{2}$ is a constant.

$$\sum_{r=2}^{\infty} \frac{1}{r^3 - r} = \lim_{n\to\infty}\sum_{r=2}^{n} \frac{1}{r^3 - r} = \lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0 + 0\right) = \frac{1}{4}$$

**4** **(i)** $f(x) = f(x+4)$

$\Rightarrow f(5) = f(1+4) = f(1)$

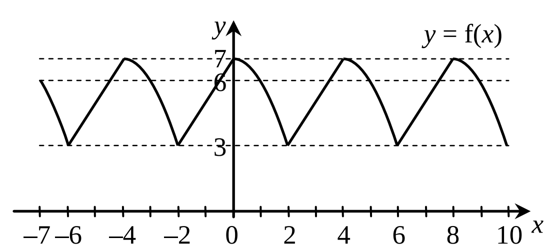
$\Rightarrow f(9) = f(5+4) = f(5)$

$f(1) = f(5) = f(9) = \ldots = f(45)$

Similarly, $f(3) = f(7) = f(11) = \ldots = f(27)$

$\therefore\ f(27) + f(45) = f(3) + f(1) = 5 + 6 = 11$

**(ii)**



**(iii)**

$$\int_{-4}^{3} \mathrm{f}(x)\,\mathrm{d}x$$
$$= \int_{-4}^{4} \mathrm{f}(x)\,\mathrm{d}x - \frac{5+7}{2}$$
$$= 2\int_{0}^{4} \mathrm{f}(x)\,\mathrm{d}x - 6$$
$$= 2\left(\int_{0}^{2} 7 - x^2\,\mathrm{d}x + \int_{2}^{4} 2x - 1\,\mathrm{d}x\right) - 6$$
$$= 2\left(\left[7x - \tfrac{x^3}{3}\right]_0^2 + \left[x^2 - x\right]_2^4\right) - 6$$
$$= 2\left(11\frac{1}{3} + 10\right) - 6$$
$$= 36\frac{2}{3} \text{ square units}$$

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**5** Let $P_n$ be the statement

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

When $n = 1$,

$$\text{L.H.S.} = \sum_{r=1}^{1} r^2 = 1^2 = 1$$

$$\text{R.H.S.} = \frac{1}{6}1(1+1)(2(1)+1) = \frac{1}{6}(1)(2)(3) = 1$$

$P_1$ is true since L.H.S. = R.H.S.

Suppose that $P_k$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

To show that $P_{k+1}$ is true, i.e.

$$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$\text{L.H.S} = \sum_{r=1}^{k+1} r^2$$
$$= \sum_{r=1}^{k} r^2 + (k+1)^2$$
$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$
$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$
$$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$$
$$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

Since $P_1$ is true, $P_{k+1}$ is true whenever $P_k$ is true, by mathematical induction, $P_n$ is true for all $n \in \mathbb{Z}^+$.

$$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$$
$$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}n(n+1)(2n+1)$$
$$= \frac{1}{6}n(2n+1)\left[2(4n+1) - (n+1)\right]$$
$$= \frac{1}{6}n(2n+1)(8n+2-n-1)$$
$$= \frac{1}{6}n(2n+1)(7n+1)$$

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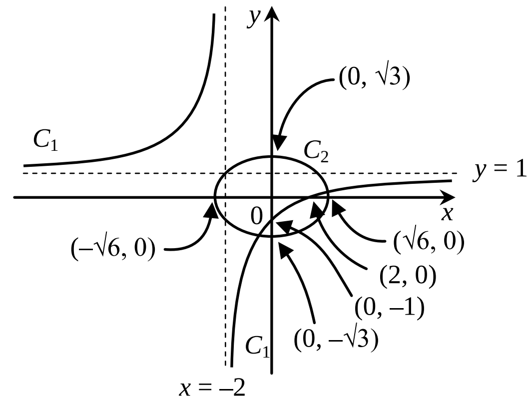
**6** **(i)**




**(ii)**

$$C_1 : y = \frac{x-2}{x+2} \Rightarrow y^2 = \left(\frac{x-2}{x+2}\right)^2$$

$$C_2 : \frac{x^2}{6} + \frac{y^2}{3} = 1 \Rightarrow y^2 = \frac{1}{2}(6 - x^2)$$

$$y^2 = \frac{(x-2)^2}{(x+2)^2} = \frac{1}{2}(6 - x^2)$$

$$2(x-2)^2 = (x+2)^2(6 - x^2)$$

**(iii)** $x = -0.515$ or $x = 2.45$

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**7** **(i)**

$$\mathrm{f}(x) = \mathrm{e}^{\cos x}$$
$$\Rightarrow \mathrm{f}(0) = \mathrm{e}^{\cos 0} = \mathrm{e}$$
$$\mathrm{f}'(x) = -\sin x \cdot \mathrm{e}^{\cos x}$$
$$\Rightarrow \mathrm{f}'(0) = -\sin 0 \cdot \mathrm{e}^{\cos 0} = 0$$
$$\mathrm{f}''(x) = -\cos x \cdot \mathrm{e}^{\cos x} + \sin^2 x \cdot \mathrm{e}^{\cos x}$$
$$\Rightarrow \mathrm{f}''(0) = -\cos 0 \cdot \mathrm{e}^{\cos 0} + \sin^2 0 \cdot \mathrm{e}^{\cos 0} = -\mathrm{e}$$

By Maclaurin's expansion,

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0)$$
$$= \mathrm{e} + 0 + \frac{x^2}{2!}(-\mathrm{e})$$
$$= \mathrm{e} - \frac{\mathrm{e}}{2}x^2$$

**(ii)**

$$\frac{1}{a+bx^2}=(a+bx^2)^{-1}$$

$$=a^{-1}\left(1+\frac{b}{a}x^2\right)^{-1}$$

$$=a^{-1}\left(1+(-1)\left(\frac{b}{a}x^2\right)+\ldots\right)$$

$$=a^{-1}\left(1-\frac{b}{a}x^2+\ldots\right)$$

$$=\frac{1}{a}-\frac{b}{a^2}x^2+\ldots$$

Comparing the term independent of $x$,

$\frac{1}{a}=\mathrm{e}\Rightarrow a=\frac{1}{\mathrm{e}}$

Comparing the coefficient of $x^2$,

$-\frac{\mathrm{e}}{2}=-\frac{b}{a^2}\Rightarrow b=\frac{a^2\mathrm{e}}{2}=\frac{1}{2\mathrm{e}}$

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**8** **(i)** Let $u_n$, $S_n$ and $r$ denote the length of the $n$th bar in cm, the total length of the first $n$ bars in cm, and the common ratio of the geometric sequence respectively.

$u_1=20$

$u_{25}=5=u_1r^{25-1}=20r^{24}$

$r^{24}=\frac{5}{20}=\frac{1}{4}\Rightarrow r=\sqrt[24]{\frac{1}{4}}$

$S_\infty=\frac{u_1}{1-r}=\frac{20}{1-\sqrt[24]{\frac{1}{4}}}=356.34<357$

Therefore, the total length of all the bars must be less than 357 cm, no matter how many bars there are.

**(ii)**

$$L=S_{25}=\frac{u_1(1-r^{25})}{1-r}=\frac{20(1-(\sqrt[24]{\frac{1}{4}})^{25})}{1-\sqrt[24]{\frac{1}{4}}}=272.26$$

$\approx 272$

The total length of all the bars of instrument $B$ is 272 cm.

$u_{13}=u_1r^{12}=20(\sqrt[24]{\frac{1}{4}})^{12}=10$

The length of the 13th bar is 10 cm.

**(iii)** Let $t_n$ and $d$ denote the length of the $n$th bar in cm and the common difference of the arithmetic sequence respectively.

$t_{25}=5$

$t_1=5-24d$

$\frac{25}{2}(t_1+t_{25})=\frac{25}{2}(5+5-24d)=272.26$

$125-300d=272.26$

$300d=125-272.26$

$\Rightarrow d=-0.491$

$\Rightarrow t_1=5-24(-0.491)\approx 16.8\text{ cm}$

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**9** **(i)** $\left|z^7\right|=\left|1+\mathrm{i}\right|=\sqrt{2}$

$\arg z^7=\arg(1+\mathrm{i})=\frac{\pi}{4}$

$z^7-(1+\mathrm{i})=0$

$z^7=1+\mathrm{i}$

$z^7=\sqrt{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)}$

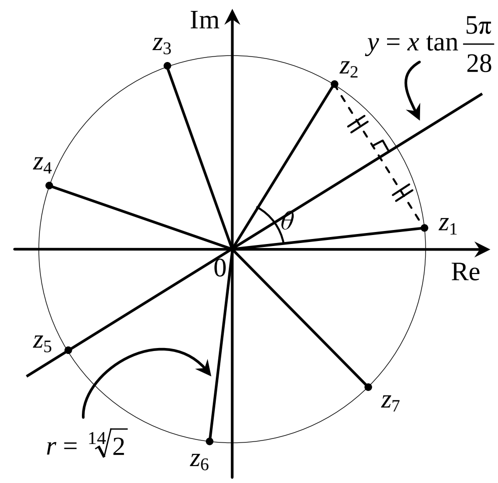
$z=\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\pi\frac{1}{7}(\frac{1}{4}+2k)}$

$z=\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{28}+\frac{2}{7}k)}$, where $k=0,\pm 1,\pm 2,\pm 3$

$\therefore z=\sqrt[14]{2}\mathrm{e}^{-\mathrm{i}\frac{23}{28}\pi},\sqrt[14]{2}\mathrm{e}^{-\mathrm{i}\frac{15}{28}\pi},\sqrt[14]{2}\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi},\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{1}{28}\pi},$

$\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{9}{28}\pi},\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{17}{28}\pi},\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{25}{28}\pi},$

**(ii)**




**(iii)** Let $z=0+0\mathrm{i}$ represent the origin.

$|z-z_1|=|z-z_2|$

$\text{L.H.S.}=\left|0+0\mathrm{i}-\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{1}{28}\pi}\right|=\sqrt[14]{2}$

$\text{R.H.S.}=\left|0+0\mathrm{i}-\sqrt[14]{2}\mathrm{e}^{\mathrm{i}\frac{9}{28}\pi}\right|=\sqrt[14]{2}$

Since L.H.S. = R.H.S., the locus of all points $z$ such that $|z-z_1|=|z-z_2|$ passes through the origin.

Angle with respect to the positive $x$-axis

$=\frac{1}{2}\left(\frac{\pi}{28}+\frac{9\pi}{28}\right)=\frac{5\pi}{28}$

Cartesian equation of the locus is $y=x\tan\frac{5\pi}{28}$.

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**10** **(i)** Let $\theta$ denote the acute angle between $p_1$ and $p_2$.

$$\cos\theta=\frac{\left|\begin{pmatrix}2\\1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\1\\3\end{pmatrix}\right|\left|\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|}$$

$$=\frac{(2)(-1)+(1)(2)+(3)(1)}{\sqrt{2^2+1^2+3^2}\sqrt{(-1)^2+2^2+1^2}}=\frac{3}{\sqrt{14}\sqrt{6}}$$

$$\theta=\cos^{-1}\left(\frac{3}{\sqrt{14}\sqrt{6}}\right)=70.893^\circ\approx 70.9^\circ$$

**(ii)** $\begin{pmatrix}2\\1\\3\end{pmatrix}\times\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}(1)(1)-(3)(2)\\(3)(-1)-(2)(1)\\(2)(2)-(1)(-1)\end{pmatrix}=\begin{pmatrix}-5\\-5\\5\end{pmatrix}=5\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\left.\begin{array}{l}p_1: 2x+y+3z=1\\ p_2: -x+2y+z=2\end{array}\right\}\quad x=-z,\ y=1-z$

When $z=0$, $x=0$ and $y=1$

$\therefore l:\mathbf{r}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \lambda\in\mathbb{R}$

**(iii)** $p_3: 2x+y+3z-1+k(-x+2y+z-2)=0$

$\Leftrightarrow (2-k)x+(1+2k)y+(3+k)z=1+2k$

$\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2-k\\1+2k\\3+k\end{pmatrix}=1+2k\Rightarrow l \text{ contains }\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\begin{pmatrix}-1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}2-k\\1+2k\\3+k\end{pmatrix}=-(2-k)-(1+2k)+(3+k)=0$

$\Rightarrow$ The direction vector of $l$ is parallel to $p_3$.

$\therefore l$ lies in $p_3$ for any constant $k$.

$\begin{pmatrix}2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}2-k\\1+2k\\3+k\end{pmatrix}=2k+1$

$2(2-k)+3(1+2k)+4(3+k)=2k+1$

$4-2k+3+6k+12+4k=2k+1$

$8k+19=2k+1$

$\Rightarrow k=-3$

$\therefore\ p:\mathbf{r}\cdot\begin{pmatrix}2-(-3)\\1+2(-3)\\3+(-3)\end{pmatrix}=\mathbf{r}\cdot\begin{pmatrix}5\\-5\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-5\\0\end{pmatrix}=-5$

$\Rightarrow 5x-5y=-5$

$\therefore x-y=-1, z\in\mathbb{R}$

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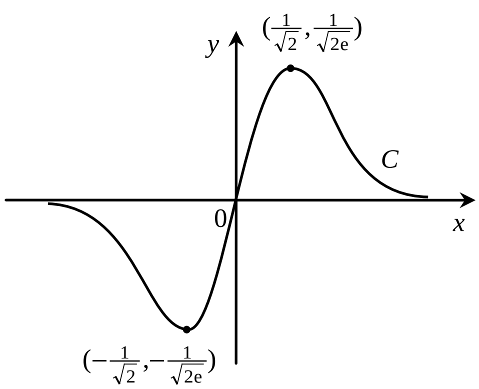
**11** **(i)**




**(ii)** $f(x)=xe^{-x^2}$

$f'(x)=e^{-x^2}+xe^{-x^2}(-2x)=e^{-x^2}-2x^2e^{-x^2}$

$e^{-x^2}-2x^2e^{-x^2}=0$

$e^{-x^2}=2x^2e^{-x^2}$

$x^2=\frac{1}{2}\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

When $x=\frac{1}{\sqrt{2}}$, $y=\frac{1}{\sqrt{2}}e^{-\frac{\sqrt{2}}{2}^2}=\frac{1}{\sqrt{2e}}$

When $x=-\frac{1}{\sqrt{2}}$, $y=-\frac{1}{\sqrt{2}}e^{-\left(-\frac{\sqrt{2}}{2}\right)^2}=-\frac{1}{\sqrt{2e}}$

Coordinates of the turning points are

$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2e}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2e}}\right)$.

**(iii)** $x=0\Rightarrow u=0$

$x=n\Rightarrow u=n^2$

$u=x^2\Rightarrow x=\sqrt{u}$

$\int_0^n f(x)\,dx=\int_0^n xe^{-x^2}\,dx$

$=\int_0^{n^2}\sqrt{u}e^{-u}\frac{dx}{du}du$

$=\int_0^{n^2}\sqrt{u}e^{-u}\frac{1}{2\sqrt{u}}du$

$=\frac{1}{2}\int_0^{n^2}e^{-u}\,du$

$=\frac{1}{2}\left[-e^{-u}\right]_0^{n^2}$

$=\frac{1}{2}(-e^{-n^2}+e^0)$

$=\frac{1}{2}(1-e^{-n^2})$

Area of region between $C$ and the positive $x$ - axis

$=\lim_{n\to\infty}\int_0^n f(x)\,dx=\frac{1}{2}(1-0)=\frac{1}{2}$ square units

**(iv)** $\int_{-2}^{2}|f(x)|\,dx=2\int_0^2 f(x)\,dx=2\times\frac{1}{2}(1-e^{-2^2})$

$=1-e^{-4}$

**(v)** Volume of revolution

$=\pi\int_0^1(f(x))^2\,dx$

$=\pi\int_0^1(xe^{-x^2})^2\,dx$

$=0.11570\pi$

$=0.36349$

$\approx 0.363$ cubic units

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